Gradient, Divergence and Curl
in Curvilinear Coordinates

Although cartesian orthogonal coordinates are very intuitive and easy to use, it is often found more convenient to work with other coordinate systems. Being able to change all variables and expression involved in a given problem, when a different coordinate system is chosen, is one of those skills a physicist, and even more a theoretical physicist, needs to possess.

In this lecture a general method to express any variable and expression in an arbitrary curvilinear coordinate system will be introduced and explained. We will be mainly interested to find out general expressions for the gradient, the divergence and the curl of scalar and vector fields. Specific applications to the widely used cylindrical and spherical systems will conclude this lecture.

1   The concept of orthogonal curvilinear coordinates

The cartesian orthogonal coordinate system is very intuitive and easy to handle. Once an origin has been fixed in space and three orthogonal scaled axis are anchored to this origin, any point in space is uniquely determined by three real numbers, its cartesian coordinates. A curvilinear coordinate system can be defined starting from the orthogonal cartesian one. If x, y, z are the cartesian coordinates, the curvilinear ones, u, v, w, can be expressed as smooth functions of x, y, z, according to:

$$
\begin{aligned}
u &= u(x,y,z) \\
v &= v(x,y,z) \\
w &= w(x,y,z)
\end{aligned}
\tag{1}
$$

These functions can be inverted to give x, y, z-dependency on u, v, w:

$$
\begin{aligned}
x &= x(u,v,w) \\
y &= y(u,v,w) \\
z &= z(u,v,w)
\end{aligned}
\tag{2}
$$

There are infinitely many curvilinear systems that can be defined using equations (1) and (2). We are mostly interested in the so-called *orthogonal curvilinear coordinate systems*, defined as follows. Any point in space is determined by the intersection of three "warped" planes:

$$
u = \text{const} \quad , \quad v = \text{const} \quad , \quad w = \text{const}
$$

We could call these surfaces as *coordinate surfaces*. Three curves, called *coordinate curves*, are formed by the intersection of pairs of these surfaces. Accordingly, three straight lines can be calculated as tangent lines to each coordinate curve at the space point. In an orthogonal curved system these three tangents will be orthogonal for all points in space (see Figure 1).

In order to express differential operators, like the gradient or the divergence, in curvilinear coordinates it is convenient to start from the infinitesimal increment in cartesian coordinates,
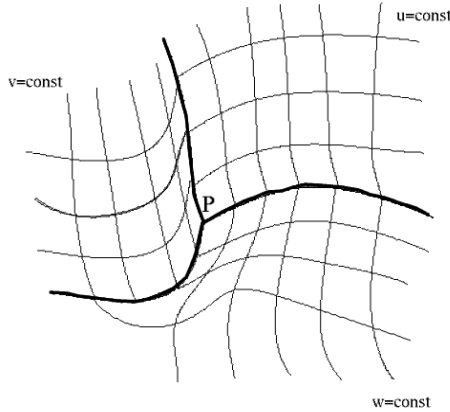
Figure 1: In this generic orthogonal curved coordinate system three coordinate surfaces meet at each point P in space. Their mutual intersection gives rise to three coordinate curves which are themselves perpendicular in P.

$d\mathbf{r} \equiv (dx, dy, dz)$. By considering equations (2) and expanding the differential $d\mathbf{r}$, the following equation can be obtained:

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u} du + \frac{\partial \mathbf{r}}{\partial v} dv + \frac{\partial \mathbf{r}}{\partial w} dw \tag{3}$$

$\partial \mathbf{r}/\partial u$, $\partial \mathbf{r}/\partial v$ and $\partial \mathbf{r}/\partial w$ are vectors tangent, respectively, to coordinate curves along u, v and w, in P. These vectors are mutually orthogonal, because we are working with orthogonal curvilinear coordinates. Let us call $\mathbf{e}_u$, $\mathbf{e}_v$ and $\mathbf{e}_w$, unit-length vectors along $\partial \mathbf{r}/\partial u$, $\partial \mathbf{r}/\partial v$ and $\partial \mathbf{r}/\partial w$, respectively. If we define by h_u, h_v and h_w as:

$$h_u \equiv \left| \frac{\partial \mathbf{r}}{\partial u} \right| \quad , \quad h_v \equiv \left| \frac{\partial \mathbf{r}}{\partial v} \right| \quad , \quad h_w \equiv \left| \frac{\partial \mathbf{r}}{\partial w} \right| \tag{4}$$

then the infinitesimal increment (3) can be re-written as:

$$d\mathbf{r} = h_u du \, \mathbf{e}_u + h_v dv \, \mathbf{e}_v + h_w dw \, \mathbf{e}_w \tag{5}$$

Equation (5), and associated defnitions (4), are instrumental in the derivation of many fundamental quantities used in differential calculus, when passing from a cartesian to a curvilinear coordinate system. Let us consider, for example, polar coordinates, (r, θ), in the plane. x and y are functions of r and θ according to:

$$\begin{cases} x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{cases}$$

To derive the correct expression for $d\mathbf{r} \equiv (dx, dy)$ we need first to compute h_r and h_θ. From (4) we get:

$$h_r = \left| \left(\frac{\partial x}{\partial r}, \frac{\partial y}{\partial r} \right) \right| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1$$

$$h_\theta = \left| \left(\frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta} \right) \right| = \sqrt{[-r\sin(\theta)]^2 + [r\cos(\theta)]^2} = r$$

Thus, $d\mathbf{r}$ is given by:

$$d\mathbf{r} = dr \, \mathbf{e}_r + r d\theta \, \mathbf{e}_\theta$$

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With this result we are able to derive the form of several quantities in polar coordinates. For example, the line element is given by:

$$d\ell \equiv \sqrt{d\mathbf{r} \cdot d\mathbf{r}} = \sqrt{(dr)^2 + r^2(d\theta)^2}$$

while the area element is:

$$dS = h_r h_\theta dr d\theta = r dr d\theta$$

For the general, 3D, case the line element is given by:

$$d\ell \equiv \sqrt{d\mathbf{r} \cdot d\mathbf{r}} = \sqrt{(h_u du)^2 + (h_v dv)^2 + (h_w dw)^2} \tag{6}$$

and the volume element is:

$$dV \equiv [(\mathbf{e}_u \cdot d\mathbf{r})\mathbf{e}_u] \cdot \{[(\mathbf{e}_v \cdot d\mathbf{r})\mathbf{e}_v] \times [(\mathbf{e}_w \cdot d\mathbf{r})\mathbf{e}_w]\} = h_u h_v h_w du dv dw \tag{7}$$

For the curl computation it is also important to have ready expressions for the surface elements perpendicular to each coordinate curve. These elements are simply given as:

$$dS_u = h_v h_w dv dw \quad , \quad dS_v = h_u h_w du dw \quad , \quad dS_w = h_u h_v du dv \tag{8}$$

2 Gradient in curvilinear coordinates

Given a function $f(u, v, w)$ in a curvilinear coordinate system, we would like to find a form for the gradient operator. In order to do so it is convenient to start from the expression for the function differential. We have either,

$$df = \nabla f \cdot d\mathbf{r} \tag{9}$$

or, expanding df using curvilinear coordinates,

$$df = \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv + \frac{\partial f}{\partial w}dw$$

Let us now multiply and divide each term in the summation by three different functions:

$$df = \frac{1}{h_u}\frac{\partial f}{\partial u}h_u du + \frac{1}{h_v}\frac{\partial f}{\partial v}h_v dv + \frac{1}{h_w}\frac{\partial f}{\partial w}h_w dw$$

Using (5) the following identities can immediately be derived:

$$\mathbf{e}_u \cdot d\mathbf{r} = h_u du \quad , \quad \mathbf{e}_v \cdot d\mathbf{r} = h_v dv \quad , \quad \mathbf{e}_w \cdot d\mathbf{r} = h_w dw$$

Thus, we can re-write df as:

$$df = \frac{1}{h_u}\frac{\partial f}{\partial u}\mathbf{e}_u \cdot d\mathbf{r} + \frac{1}{h_v}\frac{\partial f}{\partial v}\mathbf{e}_v \cdot d\mathbf{r} + \frac{1}{h_w}\frac{\partial f}{\partial w}\mathbf{e}_w \cdot d\mathbf{r} = \left(\frac{1}{h_u}\frac{\partial f}{\partial u}\mathbf{e}_u + \frac{1}{h_v}\frac{\partial f}{\partial v}\mathbf{e}_v + \frac{1}{h_w}\frac{\partial f}{\partial w}\mathbf{e}_w\right) \cdot d\mathbf{r}$$

Comparing this last result with (9) we finally get:

$$\nabla f = \frac{1}{h_u}\frac{\partial f}{\partial u}\mathbf{e}_u + \frac{1}{h_v}\frac{\partial f}{\partial v}\mathbf{e}_v + \frac{1}{h_w}\frac{\partial f}{\partial w}\mathbf{e}_w \tag{10}$$

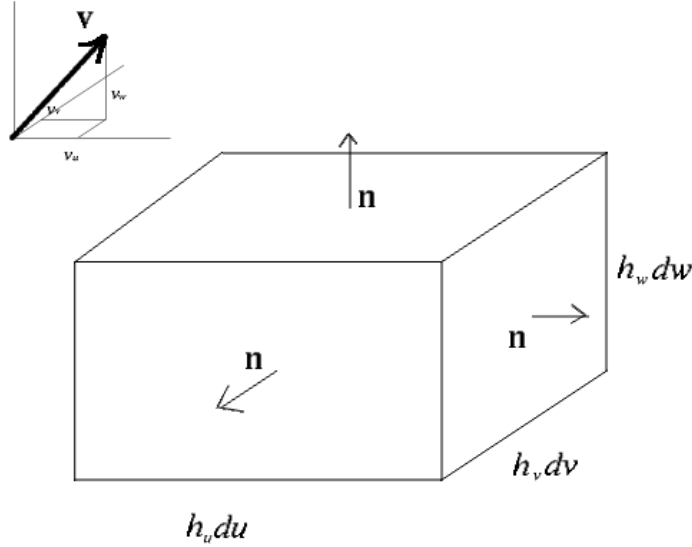which is the general form for the gradient in curvilinear coordinates, we were looking for.

Figure 2: Volume element in curvilinear coordinates. The sides of the small parallelepiped are given by the components of d**r** in equation (5). Vector **v** is decomposed into its u-, v- and w-components.

3   Divergence and laplacian in curvilinear coordinates

Consider a volume element around a point P with curvilinear coordinates (u, v, w). The divergence of a vector field **v** at P is defined as:

$$\lim_{\Delta V \to 0} \frac{1}{\Delta V} \oint_S \mathbf{v} \cdot \mathbf{n} \mathrm{d}S \tag{11}$$

where S is the closed surface surrounding the volume element whose volume is ΔV, and **n** is the outward-pointing normal associated to the closed surface S (see Figure 2). To compute the surface integral in equation (11) we simply have to project vector **v** along its u-, v- and w-components, and multiply the result by each perpendicular area element, in turn. In Figure 2 point P is hidden in the centre of the parallelepiped. This means that P is placed at coordinates (u, v, w), and it is half-way between $u - \mathrm{d}u/2$ and $u + \mathrm{d}u/2$ along u, half-way between $v - \mathrm{d}v/2$ and $v + \mathrm{d}v/2$ along v, and half-way between $w - \mathrm{d}w/2$ and $w + \mathrm{d}w/2$ along w. Let us compute the contribution to the integral from v_u first. This is equivalent to

$$-v_u h_v h_w \mathrm{d}v \mathrm{d}w$$

where v_u, h_v and h_w are computed at $u - \mathrm{d}u/2$, summed to

$$v_u h_v h_w \mathrm{d}v \mathrm{d}w$$

where v_u, h_v and h_w are computed at $u + \mathrm{d}u/2$. Using Taylor expansion yields the following contribution for the v_u component of the field:

$$\int_{\text{surf} \perp \text{to } u} \mathbf{v} \cdot \mathbf{n} \mathrm{d}S = \frac{\partial(v_u h_v h_w)}{\partial u} \mathrm{d}u \mathrm{d}v \mathrm{d}w \tag{12}$$

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Analogous expressions are obtained considering those contribution to the integral along v and w:

$$\int_{\text{surf} \perp \text{to } v} \mathbf{v} \cdot \mathbf{n} \mathrm{d}S = \frac{\partial(h_u v_v h_w)}{\partial v} \mathrm{d}u \mathrm{d}v \mathrm{d}w \tag{13}$$

$$\int_{\text{surf} \perp \text{to } w} \mathbf{v} \cdot \mathbf{n} \mathrm{d}S = \frac{\partial(h_u h_v v_w)}{\partial w} \mathrm{d}u \mathrm{d}v \mathrm{d}w \tag{14}$$

Thus, the closed-surface integral is given by adding up (12), (13) and (14):

$$\oint_S \mathbf{v} \cdot \mathbf{n} \mathrm{d}S = \left[\frac{\partial(v_u h_v h_w)}{\partial u} + \frac{\partial(h_u v_v h_w)}{\partial v} + \frac{\partial(h_u h_v v_w)}{\partial w} \right] \mathrm{d}u \mathrm{d}v \mathrm{d}w$$

Using $\mathrm{d}V = h_u h_v h_w \mathrm{d}u \mathrm{d}v \mathrm{d}w$, we arrive, eventually, at the final expression for the divergence in curvilinear coordinates replacing last quantity into formula (11):

$$\nabla \cdot \mathbf{v} = \frac{1}{h_u h_v h_w} \left[\frac{\partial(v_u h_v h_w)}{\partial u} + \frac{\partial(h_u v_v h_w)}{\partial v} + \frac{\partial(h_u h_v v_w)}{\partial w} \right] \tag{15}$$

From this last equation is also very immediate to derive the expression for the laplacian of a scalar field φ. This is defined as:

$$\nabla^2 \varphi \equiv \nabla \cdot (\nabla \varphi)$$

Thus we only need to link formulas (10) and (15) together

$$\nabla^2 \varphi = \frac{1}{h_u h_v h_w} \left[\frac{\partial}{\partial u} \left(\frac{1}{h_u} \frac{\partial \varphi}{\partial u} h_v h_w \right) + \frac{\partial}{\partial v} \left(h_u \frac{1}{h_v} \frac{\partial \varphi}{\partial v} h_w \right) + \frac{\partial}{\partial w} \left(h_u h_v \frac{1}{h_w} \frac{\partial \varphi}{\partial w} \right) \right]$$

to yield, eventually,

$$\nabla^2 \varphi = \frac{1}{h_u h_v h_w} \left[\frac{\partial}{\partial u} \left(\frac{h_v h_w}{h_u} \frac{\partial \varphi}{\partial u} \right) + \frac{\partial}{\partial v} \left(\frac{h_u h_w}{h_v} \frac{\partial \varphi}{\partial v} \right) + \frac{\partial}{\partial w} \left(\frac{h_u h_v}{h_w} \frac{\partial \varphi}{\partial w} \right) \right] \tag{16}$$

4   Curl in curvilinear coordinates

The curl of a vector field is another vector field. Its component along an arbitrary vector $\mathbf{n}$ is given by the following expression:

$$[\nabla \times \mathbf{v}]_n \equiv \lim_{\Delta S \to 0} \frac{1}{\Delta S} \oint_\gamma \mathbf{v} \cdot \mathrm{d}\mathbf{r} \tag{17}$$

where γ is a curve encircling the small area element ΔS, and $\mathbf{n}$ is perpendicular to ΔS. Let us start with the w-component. We need to select a surface element perpendicular to $\mathbf{e}_w$. This is given in Figure 3. The contribution to the line integral coming from segments 1 and 3 are

$$v_u h_u \mathrm{d}u$$

computed at $v - \mathrm{d}v/2$, and

$$-v_u h_u \mathrm{d}u$$

computed at $v + \mathrm{d}v/2$. These, added together, gives:

$$-\frac{\partial(h_u v_u)}{\partial v} \mathrm{d}u \mathrm{d}v \tag{18}$$

The contribution from segments 2 and 4 gives, on the other hand,
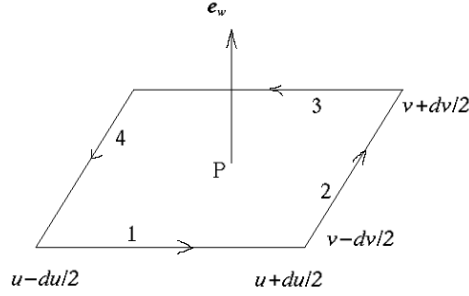
$$v_v h_v \mathrm{d}v$$

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Figure 3: Surface element for the determination of curl's component along w, in curvilinear coordinates.

computed at $u + \mathrm{d}u/2$, and

$$-v_v h_v \mathrm{d}v$$

computed at $u - \mathrm{d}u/2$. Adding them together yields

$$\frac{\partial(h_v v_v)}{\partial u}\mathrm{d}u\mathrm{d}v \qquad (19)$$

From the partial results (18) and (19) we obtain the contribution to the curl we were looking for:

$$[\nabla \times \mathbf{v}]_{\mathbf{e}_w} = \frac{1}{h_u h_v \mathrm{d}u\mathrm{d}v}\left[\frac{\partial(h_v v_v)}{\partial u} - \frac{\partial(h_u v_u)}{\partial v}\right]\mathrm{d}u\mathrm{d}v = \frac{1}{h_u h_v}\left[\frac{\partial(h_v v_v)}{\partial u} - \frac{\partial(h_u v_u)}{\partial v}\right]$$

The other two components can be derived from the previous expression with the cyclic permutation $u \to v \to w \to u$. To extract all three components the following compressed determinantal form can be used:

$$\nabla \times \mathbf{v} = \frac{1}{h_u h_v h_w}\begin{vmatrix} h_u \mathbf{e}_u & h_v \mathbf{e}_v & h_w \mathbf{e}_w \\ \partial/\partial u & \partial/\partial v & \partial/\partial w \\ h_u v_u & h_v v_v & h_w v_w \end{vmatrix} \qquad (20)$$

In the final section we will derive expressions for the laplacian in cylindrical and spherical coordinates, given the importance played by Laplace's equation in Mathematical Physics.

5 Laplacian in cylindrical and spherical coordinates

a) CYLINDRICAL COORDINATES
Cylindrical coordinates, (r, θ, z), are related to cartesian ones through the following transformation:

$$\begin{cases} x &= r\cos(\theta) \\ y &= r\sin(\theta) \\ z &= z \end{cases} \qquad (21)$$

Using definition (4) we get:

$$h_r = \sqrt{(\partial x/\partial r)^2 + (\partial y/\partial r)^2 + (\partial z/\partial r)^2} = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1$$

$$h_\theta = \sqrt{(\partial x/\partial \theta)^2 + (\partial y/\partial \theta)^2 + (\partial z/\partial \theta)^2} = \sqrt{r^2 \cos^2(\theta) + r^2 \sin^2(\theta)} = r$$

$$h_z = \sqrt{(\partial x/\partial z)^2 + (\partial y/\partial z)^2 + (\partial z/\partial z)^2} = \sqrt{1} = 1$$

Replacing these values in expression (16), yields:

$$\nabla^2 \varphi = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial \varphi}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial \varphi}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial \varphi}{\partial z}\right)\right]$$

or, simplifying:

$$\nabla^2 \varphi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \varphi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2} + \frac{\partial^2 \varphi}{\partial z^2} \tag{22}$$

b) SPHERICAL COORDINATES

Spherical coordinates, (r, θ, ϕ), are related to cartesian coordinates through:

$$\begin{cases} x &=& r\sin(\theta)\cos(\phi) \\ y &=& r\cos(\theta)\sin(\phi) \\ z &=& r\cos(\theta) \end{cases} \tag{23}$$

Using definition (4) we get:

$$h_r = \sqrt{(\partial x/\partial r)^2 + (\partial y/\partial r)^2 + (\partial z/\partial r)^2} = \sqrt{\sin^2(\theta)\cos^2(\phi) + \sin^2(\theta)\cos^2(\phi) + \cos^2(\theta)} = 1$$

$$h_\theta = \sqrt{(\partial x/\partial \theta)^2 + (\partial y/\partial \theta)^2 + (\partial z/\partial \theta)^2} = \sqrt{r^2\cos^2(\theta)\cos^2(\phi) + r^2\cos^2(\theta)\cos^2(\phi) + r^2\sin^2(\theta)} = r$$

$$h_\phi = \sqrt{(\partial x/\partial \phi)^2 + (\partial y/\partial \phi)^2 + (\partial z/\partial \phi)^2} = \sqrt{r^2\sin^2(\theta)\sin^2(\phi) + r^2\sin^2(\theta)\cos^2(\phi)} = r\sin(\theta)$$

Replacing these values in expression (16), yields:

$$\nabla^2 \varphi = \frac{1}{r^2\sin(\theta)}\left\{ \frac{\partial}{\partial r}\left[r^2\sin(\theta)\frac{\partial \varphi}{\partial r}\right] + \frac{\partial}{\partial \theta}\left[\sin(\theta)\frac{\partial \varphi}{\partial \theta}\right] + \frac{\partial}{\partial \phi}\left[\frac{1}{\sin(\theta)}\frac{\partial \varphi}{\partial \phi}\right]\right\}$$

or, simplifying:

$$\nabla^2 \varphi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \varphi}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial \theta}\left[\sin(\theta)\frac{\partial \varphi}{\partial \theta}\right] + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2 \varphi}{\partial \phi^2} \tag{24}$$